

Spectral multiplicity for powers of weakly mixing automorphisms

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Abstract

We study the behavior of maximal multiplicities $mm(R^n)$ for the powers of a weakly mixing automorphism R . For some special infinite set A we show the existence of a weakly mixing rank-one automorphism R such that $mm(R^n) = n$ and $mm(R^{n+1}) = 1$ for all $n \in A$. Moreover, the cardinality $cardm(R^n)$ of the set of spectral multiplicities for R^n is not bounded. We have $cardm(R^{n+1}) = 1$ and $cardm(R^n) = 2^{m(n)}$, $m(n) \rightarrow \infty$, $n \in A$. We also construct another weakly mixing automorphism R with the following properties: $mm(R^n) = n$ for $n = 1, 2, 3, \dots, 2009, 2010$ but $mm(R^{2011}) = 1$, all powers (R^n) have homogeneous spectrum, and the set of limit points of the sequence $\{\frac{mm(R^n)}{n} : n \in \mathbf{N}\}$ is infinite.

1 Introduction

We recall the notion of maximal spectral multiplicity $mm(U)$ of a unitary operator U . Consider all representations of the form

$$U \equiv W \oplus V \oplus V \oplus \dots \oplus V \dots$$

The maximal number of such copies of V is denoted by $mm(U)$. An automorphism T acting on a measure space (X, μ) , $\mu(X) = 1$, induces a unitary operator $Uf = fT$ (acting on $L_2(X, \mu)$, $f \in L_2$). We write $mm(T)$ instead of $mm(U)$. The property of an automorphism to be weakly mixing means that its spectral measure is continuous. Weak mixing can also be defined as the existence of a (mixing) sequence m_i such that for all $f, g \in L_2(X, \mu)$

$$\int f U^{m_i} g d\mu \rightarrow \int f d\mu \int g d\mu.$$

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In ergodic theory there are examples of weakly mixing transformations with the following properties:

- (1) $mm(T^n) = 1, n = 1, 2, 3, 4, \dots;$
- (2) $mm(T^n) = m, n = 1, 2, 3, 4, \dots;$
- (3) $mm(T^n) = 2mn, n = 1, 2, 3, 4, \dots$

For example, the latter follows from the results of Ageev [Ag1] and Lemanczyk [L] on realization of a Lebesgue component of multiplicity 2 (here $mm(T) = 2$ implies that $mm(T^2) = 4, mm(T^3) = 6, \dots$). Property (1) is known to hold for

Gaussian systems with simple spectrum,
Ornstein's stochastic constructions [Ab],
some mixing staircase constructions [R].

The aim of this paper is to give examples of unusual behavior of $mm(T^n)$.

Theorem. *One can find an infinite set A of integers and a weakly mixing automorphism R such that $mm(R^n) = n$ and $mm(R^{n+1}) = 1$ for all $n \in A$. Moreover, the cardinalities of the multiplicity sets for R^n are not bounded. There is a weakly mixing automorphism S such that all the powers S^n have homogeneous spectrum and the set of limit points of the sequence $\{\frac{mm(R^n)}{n}\}$ is infinite.*

We *a posteriori* come to the following principle: *given ergodic automorphism H with discrete spectrum there is a weakly mixing automorphism R such that $mm(R^n) = mm(H^n)$ for all $n > 0$.* We apply Ageev's approach based on generic group actions and we control the spectrum via weak limits of powers. The latter has an old history (A.Katok[K], V.I.Oseledets[Os], A.M.Stepin [S] et al). This note mostly uses [R], [Ry].

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2 Group actions, homogeneous spectra, little joinings

Consider a measure-preserving free action of the group G_p generated by two elements s, ϕ and the following relations:

1. $\phi^p = e$,
2. the elements $s, \phi s \phi^{-1}, \phi^2 s \phi^{-2}, \dots, \phi^{p-1} s \phi^{-p+1}$ commute.

Denoting the corresponding elements of the action by S and Φ , we now put

$$R := S\Phi, \quad T := R^p.$$

It is proved in [Ry] that the automorphism R has simple spectrum for a generic G_p -action. The automorphism T and all of its positive powers have homogeneous spectrum of multiplicity p . The proof of Theorem 1.1 in [Ry] actually shows that all the automorphisms R^n have simple spectrum for $n \perp p$ (no common divisors). We must only check that Model 2 [Ry] guarantees the simplicity of spectrum of R^n . The automorphism S also has simple spectrum for a generic group action.

Factors and joinings. Our T has a factor \tilde{T} . Indeed, the automorphism T commutes with periodic Φ , we have an T -invariant algebra \mathcal{F} of Φ -invariant sets. This factor has simple spectrum [Ry], but $\mathcal{F} \neq R\mathcal{F}$ (or $\mathcal{F} \neq S\mathcal{F}$). So there is a non-trivial self-joining of this factor. The correspondence between joinings and factors is well known (see [R1], [Th] for joinings and their relations to factors). We put

$$\nu(A \times B) = \int_X \chi_A \chi_{RA} d\mu$$

for $A, B \in \mathcal{F}$. Since R commutes with T , we have

$$\nu(\tilde{T}A \times \tilde{T}B) = \int_X T\chi_A T\chi_{RA} d\mu = \int_X \chi_{\tilde{T}A} \chi_{R\tilde{T}A} d\mu = \nu(A \times B).$$

Thus we have self-joinings of \tilde{T} , but the measure ν is well defined for all measurable sets A, B . In the theory of joinings, this measure is said to be off-diagonal. The action of $T \times T$ with this invariant measure is naturally isomorphic to T via the map $x \mapsto (x, Rx)$. We take $p = 3$ and consider the map $x \mapsto (Rx, R\Phi x, R\Phi^2 x)$. There is a set D such that X is a disjoint

union of $D, \Phi D, \Phi^2 D$. Our factor \tilde{T} is associated with the map

$$\tilde{T} : D \rightarrow D, \quad \tilde{T}x = \Phi^{n(x)}Tx,$$

where $n(x)$ is defined by the condition $\Phi^{n(x)}Tx \in D$. We consider the map

$$x \mapsto (\Phi^{n_1(x)}Rx, \Phi^{n_2(x)}R\Phi x, \Phi^{n_3(x)}R\Phi^2x) \in D \times D \times D.$$

Since the relation $\Phi^n R \Phi^m = R$ implies in our group that $n = m = 0(\text{mod } p)$, we see that the points $\Phi^{n_1(x)}Rx, \Phi^{n_2(x)}R\Phi x, \Phi^{n_3(x)}R\Phi^2x$ must be different for almost all $x \in D$. (Here we use the obvious fact that different automorphisms commuting with an ergodic action have pairwise disjoint graphs.) So we obtain a bijection between the off-diagonal in $X \times X$ and the graph of this 3-valued map on D . Thus our joining sits on the graph of a 3-valued map.

Remarks. 1. If \tilde{T} has simple spectrum and commutes with an n -valued map, then the adjoint map is also n -valued. Indeed, the corresponding commuting Markov operators J and J^* satisfy

$$\frac{1}{m}I + \frac{m-1}{m}Q = J^*J = JJ^* = \frac{1}{n}I + \frac{n-1}{n}Q',$$

where the graphs of Q and Q' are disjoint from the diagonal. Thus, $\frac{1}{m}I = \frac{1}{n}I$, $m = n$.

2. For a generic G_p -action, the corresponding \tilde{T} is of rank one. Moreover, it is a so-called flat-stack rank-one automorphism. J. King proved that each ergodic joining of such an automorphism is a weak limit of off-diagonal measures ($\tilde{T}^{n_i} \rightarrow^w J$).

3. S. Tikhonov used G_p -actions to prove the existence of a mixing automorphism T with p -homogeneous spectrum. Thus he also found a non-trivial self-joining for the mixing automorphism \tilde{T} . However, when $p > 2$, our factor \tilde{T} cannot be of rank one because of King's another well-known theorem saying that mixing rank-one automorphisms possess the property of minimal self-joinings (see [R1] for King's theorem). So it follows from $\mathcal{F} \neq R\mathcal{F}$ that these factors are independent, hence, the joining ν is a product measure. This contradicts the fact that ν is situated on a graph. In

fact the local rank of mixing automorphism T is not greater than p^{-1} (see [Th]).

Problem: *must the local ranks of mixing automorphisms R, \tilde{T} in question be zero?* We conjecture that our present spectral results can be established for mixing automorphisms via Tikhonov's approach [T].

4. It is not hard to obtain a finite-valued self-joining of a mixing map with simple spectrum. Suppose that the mixing symmetric product $T^{\odot 3}$ has simple spectrum (see [R]). Consider $T \otimes T \otimes T$ and two factors: the symmetric factor \mathcal{F} and a perturbation of it, say, $(I \otimes I \otimes T)\mathcal{F}$. These two factors generate the algebra of all measurable sets. The corresponding joining of $T^{\odot 3}$ will be finite-valued. The dynamics on this joining exactly imitates $T \otimes T \otimes T$. We also note that $mm(T \otimes T \otimes T) = 3!$ and the spectrum of $T \otimes T \otimes T$ is non-homogeneous (type (3,6)).

5. Hard problems. Find

- 5.1. an ergodic T such that $mm(T^{p^n}) = 1$, $mm(T^{p^{n+1}}) = p$ for some $p, n > 1$;
- 5.2. an ergodic automorphism T such that $mm(T^n)$ is not bounded, but $mm(T^n) < \ln \ln(n) + Const$;
- 5.3. an automorphism T such that $Rank(T^{p^n}) = 1$, $Rank(T^{p^{n+1}}) = p$ for some $p, n > 1$;
- 5.4. a flow T_t such that $mm(T_s) = 1$, $0 < s < 1$, $mm(T_2) = 2$ (and variations of this problem including rank analogies).

6. *Easy problem.* Find a weakly mixing flow T_t such that $mm(T_r) = \infty$ for all rational r and $mm(T_a) = 1$ for all irrational a . We note that there is a discrete-spectrum model, so the reader could apply our principles and methods.

3 Maximal spectral multiplicity for the powers. Periodic cases

In [R] we endowed a weakly mixing automorphism T with (G_p -generic) properties: arbitrary polynomial weak limits for its powers. The results of [R],[Ry] provided all the powers T^N , $N > 0$, homogeneous spectrum of multiplicity p . We note that [A1] contains non-weakly-mixing examples

with $mm(T^n)$ of the type $(1,2,1,2,\dots)$. Weakly mixing and mixing examples were found by the author (see [R]). Here we consider $R(x, y) = (y, Tx)$ on $X \times X$. It is possible to have $mm(R^{2n}) = mm(T^n \otimes T^n) = 2$ and $mm(R^{2n+1}) = 1$.

The case $\{1, 1, \dots, 1, \mathbf{p}, 1, 1, \dots, 1, \mathbf{p}, 1, \dots\}$. We look at $mm(R^n)$ for R in a generic G_p -action. Let p be a prime. Then we obtain $mm(R^n) = 1$ for all $n \perp p$ (no common divisors). We have $mm(R^{np}) = p$.

The case $\{1, \dots, 1, \mathbf{p}, 1, \dots, 1, \mathbf{q}, 1, \dots, 1, \mathbf{pq}, 1, \dots\}$. Let p, q be primes. Then the desired behavior of $mm(R^{pqN})$ is easily seen to hold for the G_{pq} -action with $mm(R^{pqN}) = pq$. However we propose a more complicated solution to explain our strategy for the infinite (inverse) limit construction.

We fix the primes p, q and consider the transformations R_1, R_2 for the actions of G_p, G_q respectively. Put $T_1 = R_1^p, T_2 = R_2^q$. Our first aim is to find conditions that guarantee the simplicity of spectrum of $R = R_1 \times R_2$ and, moreover, we wish to have

$$\begin{aligned} mm(R^n) &= pq \text{ if } n = n'pq, n' > 0; \\ mm(R^n) &= 1 \text{ if } n \perp pq; \\ mm(R^n) &= p \text{ if } n = n'p \text{ and } n' \perp q; \\ mm(R^n) &= q \text{ if } n = n'q \text{ and } n' \perp p; \end{aligned}$$

To obtain this spectral behavior of the powers, we use the following weak limit for some sequence n_i (condition WL(2)):

$$(R_1 \otimes R_2)^{pn_i} \rightarrow I \otimes R_2.$$

This implies for any fixed integer $N > 0$ that

$$(R_1 \otimes R_2)^{Npn_i} \rightarrow I \otimes R_2^N.$$

LEMMA 1.(See Lemma 3.2 in [Ry]) *Suppose that R and R' have simple spectra. If there is a sequence n_i such that $(R \otimes R')^{n_i} \rightarrow I \otimes R'$, then $R \otimes R'$ has simple spectrum.*

So for all $N \perp pq$ we obtain from WL(2) and Lemma 1 that

$$mm(R_1^N \otimes R_2^N) = 1$$

since R_1^N and R_2^N have simple spectra. So, for example, we get

$$mm(R) = mm(R^{pq-1}) = mm(R^{pq+1}) = 1$$

and

$$mm(R^{pq}) = pq.$$

The last is a direct consequence of the following facts:

1. $R^{pq} = T_1^q \otimes T_2^p$ is a product of two automorphisms with multiplicities p and q , whence $mm(R^{pq}) \geq pq$;

2. $mm(R^{pq}) \leq pq$ since $mm(R) = 1$.

Combining these two facts, we get $mm(R^{pq}) = pq$.

Thus we must explain how to achieve WL(2). We find a sequence $n_i \rightarrow \infty$ and a related sequence \tilde{n}_i such that

$$pn_i = q\tilde{n}_i + 1.$$

As shown in [Ry], a generic G_p -action has the property

$$T_1^{n_i} \rightarrow I$$

for some subsequence of indices i . But for a generic G_q -action we have

$$T_2^{\tilde{n}_i} \rightarrow I$$

for some subsubsequence of indices i . Denoting this subsubsequence again by n_i , we get WL(2). Indeed,

$$R_1^{pn_i} = T_1^{n_i} \rightarrow I,$$

$$R_2^{pn_i} = R_2^{q\tilde{n}_i+1} = T_2^{\tilde{n}_i} R_2 \rightarrow R_2.$$

4 Conditions WL(k) imply that $mm(R^n) = n$ and $mm(R^{n+1}) = 1$ for all n of the form $n = p_1 p_2 \dots p_k$

We define condition WL(3) for a triple of $G_{p_1}, G_{p_2}, G_{p_3}$ -actions as the existence of sequences $n_i(1)$ and $n_i(2)$ such that

$$(R_1 \otimes R_2)^{p_1 n_i(1)} \rightarrow I \otimes R_2,$$

$$(R_1 \otimes R_2 \otimes R_3)^{p_1 p_2 n_i(2)} \rightarrow I \otimes I \otimes R_3.$$

Definition. Property WL(k) for a k -tuple of G_{p_1}, \dots, G_{p_k} -actions means that there are sequences $n_i(1), \dots, n_i(k)$ such that

$$(R_1 \otimes R_2)^{p_1 n_i(1)} \rightarrow I \otimes R_2,$$

...

$$(R_1 \otimes \dots \otimes R_k)^{p_1 \dots p_{k-1} n_i(k-1)} \rightarrow I \otimes \dots \otimes I \otimes R_k.$$

If all operators R_1, R_2, \dots have simple spectra and satisfy conditions WL(k), $k > 1$, then we see by induction that their finite direct products have simple spectra. This yields

$$mm(R_1 \otimes R_2 \otimes \dots) = 1.$$

Put $R = R_1 \otimes R_2 \otimes \dots$ and suppose that $N \perp p_1, p_2, \dots$. We claim that

$$mm(R^N) = 1.$$

Indeed, replacing the sequence $n_i(k)$ in conditions WL(k+1) by $N n_i(k)$, we obtain conditions WL(k+1) for the automorphisms R_1^N, R_2^N, \dots which also have simple spectra.

We now explain why

$$mm(R^{p_1 \dots p_k}) = p_1 \dots p_k.$$

Since $mm(R) = 1$, we know that

$$mm(R^{p_1 \dots p_k}) \leq p_1 \dots p_k.$$

Let

$$q(m) = \frac{p_1 p_2 p_3 \dots p_k}{p_m}.$$

We know that

$$mm(T_m^{q(m)}) = p_m.$$

Thus the representation

$$R^{p_1 \dots p_k} = T_1^{q(1)} \otimes T_2^{q(2)} \otimes \dots \otimes T_k^{q(k)} \otimes Q_k$$

yields that

$$mm(R^{p_1 \dots p_k}) \geq p_1 \dots p_k,$$

whence

$$mm(R^{p_1 \dots p_k}) = p_1 \dots p_k.$$

5 How to obtain conditions WL(k)

LEMMA 2. *Let \tilde{A} and A be infinite sets of integers. Then for every $p > 0$ there are two sequences $\tilde{n}_i(2) \in \tilde{A}$, $\tilde{n}_i(2) \in A$ and an automorphism T from a generic action of G_p such that $T^{\tilde{n}_i} \rightarrow I$ and $T^{n_i} \rightarrow I$.*

REMARK 3. *We can actually find the convergents $T^{\tilde{n}_i} \rightarrow P(T)$ and $T^{n_i} \rightarrow Q(T)$ for arbitrary admissible polynomials P, Q . (A polynomial $P(T) = \sum a_i T^i$ is said to be admissible if $a_i \geq 0$ and $\sum a_i = 1$.)*

Using Lemma 2 we find a rigid sequence $n_i(1)$ for T_1 : $T_1^{n_i(1)} \rightarrow I$. Then we find T_2 with rigid sequences $\tilde{n}_i(2)$ and $n_i(2) \subset n_i(1)$, where $n_i(2)$ is a subsequence of $n_i(1)$ and the sequence $\tilde{n}_i(2)$ is related to $n_i(1)$ by the equation

$$p_1 n_i(2) = p_2 \tilde{n}_i(2) + 1.$$

We now have

$$(R_1 \otimes R_2)^{p_1 n_i(2)} = (T_1^{n_i(2)} \otimes T_2^{\tilde{n}_i(2)} R_2) \rightarrow I \otimes R_2.$$

Then we find T_3 with rigid sequences $\tilde{n}_i(3)$ and $n_i(3) \subset n_i(2)$ such that $\tilde{n}_i(3)$ is related to $n_i(3)$ by the equation $p_1 p_2 n_i(3) = p_3 \tilde{n}_i(3) + 1$. Thus we obtain

$$(R_1 \otimes R_2 \otimes R_3)^{p_1 p_2 n_i(3)} = (T_1^{p_2 n_i(3)} \otimes T_2^{p_1 n_i(3)} \otimes T_3^{\tilde{n}_i(3)} R_3) \rightarrow (I \otimes I \otimes R_3).$$

Using induction, we find by Lemma 2 an automorphism T_k with rigid sequences $\tilde{n}_i(k)$ and $n_i(k) \subset n_i(k-1)$ such that $\tilde{n}_i(k)$ is related to $n_i(k)$

by the equation

$$p_1 p_2 \dots p_{k-1} n_i(k) = p_k \tilde{n}_i(k) + 1.$$

We get T_1, T_2, \dots for which the corresponding set R_1, R_2, \dots has properties WL(k).

Thus the product $R = R_1 \otimes R_2 \otimes R_3 \otimes \dots$ has simple spectrum. Using section 3, we obtain

THEOREM 4. *Let $P = \{p_1, p_2, p_3, \dots\}$, where $p_1 < p_2 < p_3 < \dots$ are primes. Suppose that $N = p_{k_1}^{d_1} p_{k_2}^{d_2} \dots p_{k_m}^{d_m} q$, $d_j > 0$, and $q \perp P$. Then $mm(R^N) = p_{k_1} p_{k_2} \dots p_{k_m}$ and $mm(R^q) = 1$. For example, if $p_{k+1} > p_1 p_2 \dots p_k$, then $mm(R^{p_1 p_2 \dots p_{k+1}}) = 1$.*

We note that the above results were partially given in [Ry] for finite products. The WL(k)-methods were used there to show that $mm(R^N) = p_{k_1} p_{k_2} \dots p_{k_m}$ for $N = p_{k_1} p_{k_2} \dots p_{k_m} M$ (the similar methods were also developed in the proof of Theorem 3.3 in [Ry]). We just applied this approach to infinite products.

Our next aim is to pass to homogeneous spectra for the powers. Before doing this, we note that the cardinality $cardm(R^N)$ of the set of spectral multiplicities for R^N can also be controlled by the same methods.

THEOREM 5. *Suppose that $N = p_{k_1}^{d_1} p_{k_2}^{d_2} \dots p_{k_m}^{d_m} q$, $d_j > 0$, and $q \perp P$. Then $cardm(R^N) = 2^m$ and $cardm(R^q) = 1$. For example, the multiplicity set of $R^{p_i p_j}$ is $\{1, p_i, p_j, p_i p_j\}$.*

The proof is a natural adaptation of the arguments proving Theorem 3.3 in [Ry] to the case of infinite products.

6 Homogeneous spectral multiplicities of the powers

Let R_p denote a cyclic permutation on the space $\{1, 2, \dots, p\}$ with the uniform measure. We consider

$$R = R_{p_1} \times R_{p_2} \times R_{p_3} \times \dots,$$

where $p_i \in P$ are different prime numbers. Our measure space (X, μ) is the group

$$X = \mathbf{Z}_{p_1} \times \mathbf{Z}_{p_2} \times \mathbf{Z}_{p_3} \times \dots$$

with Haar measure μ . The transformation R has rank one, so it is ergodic with simple discrete spectrum. The same holds for the powers R^n with any $n \perp P$. But if $n = p_1 p_2 \dots p_k$, then we have $mm(R^n) = n$.

We consider the following group of automorphisms of the group X :

$$A = \mathbf{Z}_{p_1-1} \otimes \mathbf{Z}_{p_2-1} \otimes \mathbf{Z}_{p_3-1} \otimes \dots$$

Suppose that $q \in Q \perp P$. Then R^q is conjugate to R . For every $q \in Q \perp P$ we find $\Psi_q \in A$ such that

$$\Psi_q^{-1} R \Psi_q = R^q.$$

We define a group A-R generated by R and all Ψ_q , $q \in Q$. Using some ideas of [A],[D], we easily obtain the following result.

Theorem 6. *Suppose that $Q \perp P$ are disjoint subsets of prime numbers and P is infinite. Then there is a rank-one weakly mixing transformation R conjugate to all the powers R^n with $n \in Q$.*

Proof. The above A-R-model (with a rank-one automorphism R of discrete spectrum) is amenable and free. We add a consideration of the Bernoulli action to obtain a weakly mixing A-R-generic automorphism R . The rank-one property is generic. We automatically obtain the conjugations in question. The proof is complete.

We return to $R = R_{p_1} \times R_{p_2} \times R_{p_3} \times \dots$. The power R^n is non-ergodic for $n = p_i N$. But all powers have homogeneous spectra. This suggests the following theorem, where we write $hm(T) = n$ if and only if T has homogeneous spectrum of multiplicity n .

THEOREM 7. *Let $P = \{p_1, p_2, \dots\}$ be an infinite set of prime numbers. Then there is a weakly mixing automorphism R such that we have $hm(R^N) = p_{k_1} p_{k_2} \dots p_{k_m}$ and $hm(R^q) = 1$ for all $N = p_{k_1}^{d_1} p_{k_2}^{d_2} \dots p_{k_m}^{d_m} q$, $d_j > 0$, and all $q \perp P$.*

Proof. We slightly generalize some ideas of [Ry]. Instead of the \mathbf{Z}_p -subaction, we consider a subaction of the group

$$\mathbf{Z}_{p_1} \oplus \mathbf{Z}_{p_2} \oplus \mathbf{Z}_{p_3} \oplus \dots$$

Let G_P be the group generated by the elements $R, F_1, F_2, F_3 \dots$ and the following relations:

1. $F_i F_j = F_j F_i, F_i^{p_i} = I,$
2. the elements $F_i^{-n} R F_i^{n-1}$ commute.

LEMMA 8. *A generic G_P -action possesses the following properties:*

1. R^q has a simple continuous spectrum if $q \perp P$.
2. $F_i^{-1} R$ has a simple spectrum.
3. The action of any infinite order group element have continuous spectrum.
4. R^{p_i} has homogeneous spectrum of multiplicity p_i .
5. $\text{Rank}(R^n) = 1$ for $n \perp P$, and $\text{Rank}(R^n) = mm(R^n)$.

Proof of Lemma 8.

Model 1. Property 1 can be realized using our representation $R = R_1 \otimes R_2 \otimes R_3 \otimes \dots$ from section 5. We only note that the product $R_1 \otimes R_2 \otimes R_3 \otimes \dots$ is simply connected and admits a natural free G_P -action. Indeed, put

$$F_i = I \otimes I \dots \otimes \Phi_i \otimes I \dots,$$

where Φ_i is taken from the G_{p_i} -action. Since the G_{p_i} -action is free, so is the G_P -action.

Model 2. To realize property 2, we consider R_i acting on $(X_i, \mu_i) = (\tilde{X}^{p_i}, \tilde{\mu}^{p_i})$ by the formula

$$R_i = F_i \circ (U_1 \otimes U_2 \otimes \dots U_{p_i}).$$

We achieve a simple spectrum for such R_i , say, taking U_k in a generic rank-one flow. Then

$$F_i^{-1} R = R_1 \otimes \dots R_{p_{i-1}} \otimes (U_1 \otimes U_2 \otimes \dots U_{p_i}) \otimes R_{p_{i+1}} \otimes \dots$$

Using the WL(k)-method, we again achieve a simple spectrum for this product. Thus the automorphism R and all cyclic coordinate permutations F_i generate a free G_P -action with property 2.

The Bernoulli action on Model 2 gives us property 3.

A generic G_P -action has all generic properties 1,2,3. They together imply property 4. Indeed, consider the \mathbf{Z}^{p_i} -subaction generated by the commuting automorphisms

$$S_1 = F_i^{-1}R, S_2 = F_i^{-2}RF_i^1, \dots, S_{p_i-1} = F_i^{-p_i+1}RF_i^{p_i-2}, S_{p_i} = RF_i.$$

The spectrum of this action is simple since the spectrum of S_1 is simple. We repeat the proof of Lemma 1.2 [Ry] almost word-by-word and obtain that the automorphism $S_1 S_2 \dots S_{p_i} = R^{p_i}$ has homogeneous spectrum of multiplicity p_i . The lemma is proved.

Theorem 7 follows from Lemma 8, which enables us to replace $mm(R^n)$ by $hm(R^n)$ in Theorem 4. The author plans to represent more details in a journal version.

Concluding remarks. 1. It remains to find R with the following properties: $hm(R^n) = n$, $n = 1, 2, 3, \dots, 2009, 2010$, $hm(T^{2011}) = 1$. To do this, we consider instead of the primes p_{k_1}, p_{k_2}, \dots their powers $p_{k_1}^{d_1}, p_{k_2}^{d_2}, \dots$ and use the following fact. If $hm(T) = 1$ and $hm(T^{p^d}) = p^d$, then $hm(T^{p^m}) = p^m$ for all $m < d$.

Of course, the author was lucky since 2011 is prime. So we consider all prime numbers and put $d_i = 1$ for $p_i = 2011$, and $d_j = 10$ for all $p_j \neq 2011$. Choosing the space

$$X = \mathbf{Z}_{2^{10}} \times \mathbf{Z}_{3^{10}} \times \mathbf{Z}_{5^{10}} \times \dots \times \mathbf{Z}_{2003^{10}} \times \mathbf{Z}_{2011} \times \mathbf{Z}_{2017^{10}} \times \dots,$$

we obtain required R with discrete spectrum and, via an effort of will as above, we transform the discrete spectrum into a continuous one.

2. There is a simpler way to get

$$hm(R^n) = n, \quad n = 1, 2, 3, \dots, 2009, 2010, \quad hm(T^{2011}) = 1.$$

Consider a generic G_n -action with $n = 2010!$ (see [Ry]). The corresponding R has the property above, but the behavior of $hm(R^n)$ is periodic. However, Tikhonov's approach enables us to make R mixing.

3. Connected problems. (a) Let $\mathcal{A}_{G_n!}$ denote the space of $G_n!$ -actions. We consider the projections $\pi_n : \mathcal{A}_{G_{(n+1)!}} \rightarrow \mathcal{A}_{G_n!}$ defined by $\pi_n \Phi = \Phi^n$, $\pi_n R = R^n$. Prove that for all n the image of π_n is residual in $\mathcal{A}_{G_n!}$.

(b) Find mixing automorphisms with the following properties:

1. $\{hm(R^n) : n \in \mathbf{N}\} = \{1, 2, 3, 4, \dots, k, k+1, \dots\}$;
2. $\{hm(R^n) : n \in \mathbf{N}\} = \{1, p, p^2, p^3, \dots, p^k, \dots\}$.

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